Differential Equations Comprehensive Exam
Spring 2019

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Consider a nonlinear oscillator $\ddot{q} + q^3 = 0$. Does the initial condition $q(0) = 2, \dot{q}(0) = 0$ correspond to a periodic solution? If yes, what is its period? (If you encounter an integral that is difficult to evaluate, no need to compute it out).

2. Consider $\dot{x} = x^{1/3}$ with $x(0) = 0$. What can you say about the value of $x(1)$?

3. Consider the system
$$\begin{cases} \dot{x} & = x(1 - 4x^2 - y^2) - \frac{1}{2}y(1 + x) \\ \dot{y} & = y(1 - 4x^2 - y^2) + 2x(1 + x) \end{cases}.$$
Show that all trajectories except the one from the origin approach the ellipse $4x^2 + y^2 = 1$ as $t \to +\infty$. (Hint: consider Lyapunov function $V = (1 - 4x^2 - y^2)^2$.

4. Consider $x = \sin t$ and $y = \cos t$ as a periodic solution to $\dot{x} = y, \dot{y} = -(x^2 + y^2)x$. What is the stability of this periodic orbit?

5. Assume that $u \in C^{2,1}((0, \pi) \times (0, \infty))$ solves
$$\begin{cases} u_t - u_{xx} = u, \;\; x \in (0, \pi), \;\; t > 0, \\ u(0, t) = u(\pi, t) = 0, \;\; t > 0, \\ u(x, 0) = f(x), \;\; for \; x \in (0, \pi), \end{cases}$$
where $f(x) \in C_0^\infty(0, \pi)$, that is $f(x)$ has compact support in $(0, \pi)$. Prove that
$$\lim_{t \to \infty} \|u(x, t) - C \sin(x)\|_{L^2_x([0,\pi])} = 0$$
for some constant C.

6. Assume that $f(x) \in C^1(\mathbf{R})$ is uniformly bounded function with bounded and continuous first order derivatives. Consider the following initial value probem
$$\begin{cases} u_t + uu_x = -u, \;\; x \in \mathbf{R}, t > 0, \\ u(x, 0) = f(x). \end{cases}$$
Determine the sufficient and necessary conditions on $f(x)$ for this problem to have a unique global smooth solution. For any C^1 solution $u(x, t)$ of this problem, prove that
$$\lim_{t \to \infty} \|u(x, t)\|_{L^\infty_x(\mathbf{R})} = 0.$$

7. For any finite constant $c \in \mathbf{R}$, prove that there is at most one solution $u \in C^2([0,1] \times [0,\infty))$ to the following problem

$$\begin{cases} u_{tt} - u_{xx} + cu_t = f(x,t), \ x \in (0,1), \ t > 0, \\ u(0,t) = u(1,t) = 0, \ t > 0 \\ u(x,0) = g(x), \ u_t(x,0) = h(x), \ for \ x \in (0,1), \end{cases}$$

8. Let $B(0,1)$ be the unit ball in $\mathbf{R^3}$ centered at the origin. Find a bounded solution to the following Dirichlet problem outside $B(0,1)$

$$\begin{cases} -\Delta u(x) = 0, \ |x| > 1, \\ u(x) = \dfrac{2}{\sqrt{7 + 4\sqrt{3}x_3}}, \ for \ |x| = 1. \end{cases}$$