Analysis Comprehensive Exam
August 26, 2016

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

Notations used throughout the exam:

For $E \subset \mathbb{R}^d$, the exterior Lebesgue measure of E is written $|E|_e$.
If E is measurable then its Lebesque measure is denoted $|E|$.
We denote the dual space of a Banach space V by V^*, i.e. V^* is the collection of all bounded linear functionals acting on V.

1. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $0 < |E| < \infty$.
   (i) For each $x \in \mathbb{R}$ and $r > 0$ define $I_r(x) = [\,x - r/2, x + r/2\,]$ and $h_r(x) = |E \cap I_r(x)|$. Prove that for a fixed $r > 0$, the function $h_r(x)$ is continuous at every $x \in \mathbb{R}$.
   (ii) Prove that there exists $r_0 > 0$ such that for each $0 < r < r_0$ there exists a closed interval $I \subset \mathbb{R}$ which satisfies $|I| = r$ and $|E \cap I| = \frac{r}{2}$.

2. Show that for $A \subset \mathbb{R}^d$, A is Lebesgue measurable if and only if for every $\epsilon > 0$ there exists a Lebesgue measurable set $E \subset \mathbb{R}^d$ such that
$$|A \triangle E|_e < \epsilon.$$

3. Let $E_1, ..., E_n$ be Lebesgue measurable subsets of $[0, 1]$ and define
$$S_q = \{x \in [\,0, 1\,] : x \text{ belongs to at least } q \text{ of the sets } E_i\}\,.$$
Show that for each $1 \leq q \leq n$, S_q is Lebesgue measurable and there exists k such that
$$\frac{q\,|S_q|}{n} \leq |E_k|.$$

4. Prove that if $f(x), xf(x) \in L^1(\mathbb{R})$ then the function
$$F(w) = \int_{\mathbb{R}} f(x) \sin(wx)\, dx$$
is defined, continuous, and differentiable at every point $w \in \mathbb{R}$. (You may wish to use the identity $\sin(\alpha) - \sin(\beta) = 2\sin(\frac{\alpha-\beta}{2})\cos(\frac{\alpha+\beta}{2})$).

5. Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. Given $y > 0$ denote $A_y := \{x \in \mathbb{R} : |f(x)| > y\}$. Prove that
$$\int_{\mathbb{R}} |f(x)|^p dx = p \int_0^\infty y^{p-1} |A_y| dy.$$

6. Let μ and ν be two σ-finite positive measures on a measurable space $(X, \mathfrak{M})$. Show that there exists a measurable function $f : X \to \mathbb{R}$ such that for each $E \in \mathfrak{M}$,
$$\int_E (1 - f)\, d\mu = \int_E f\, d\nu.$$
Does the above statement hold for every finite signed measures μ and ν?

7. Let $\mathcal{H}$ be a Hilbert space and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$. Prove that the following two statements are equivalent

   (i) There exists $C > 0$ such that for every $f \in \mathcal{H}$,

   $$\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \le C\|f\|^2.$$

   (ii) There exists $C > 0$ such that for every sequence $\{a_n\}_{n\in\mathbb{N}}$ with finitely many nonzero terms

   $$\Big\| \sum_{n=1}^{\infty} a_n f_n \Big\|^2 \le C \sum_{n=1}^{\infty} |a_n|^2.$$

8. Let V be a Banach space and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in V. For $m \in \mathbb{N}$ let

   $$W_m = \overline{\mathrm{span}}\{f_n\}_{n\neq m}.$$

   Prove that the following two statements are equivalent.

   (i) There exists $d > 0$ such that for every $m \in \mathbb{N}$,

   $$d \le \mathrm{dist}(f_m, W_m)\,.$$

   (ii) There exist $M > 0$ and a sequence $\{g_n\}_{n\in\mathbb{N}}$ in V^* such that for every $n \in \mathbb{N}$ we have $\|g_n\|_{V^*} < M$ and for each $m \in \mathbb{N}$,

   $$g_n(f_m) = \delta_{nm}.$$